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Existence of Free Energy for Models with Long-Range Random Hamiltonians

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Classical lattice systems with random Hamiltonians

$$\frac{1}{2}\sum_{x_1 \neq x_2} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}}$$

are considered, where d is the dimension, and $\epsilon(x_1, x_2)$ are independent random variables for different pairs (x_1, x_2), $E\epsilon(x_1, x_2) = 0$. It is shown that the free energy for such a system exists with probability 1 and does not depend on the boundary conditions, provided $\alpha > 1/2$.

KEY WORDS: Random interactions; random variables; long range; free energy; Hamiltonian; spin system; partial function.

1. INTRODUCTION

The anomalous magnetic properties of some metallic alloys, the so-called spin-glasses, are known to be due to the Ruderman–Kittel–Kasuya–Yosida (RKKY) spin–spin interaction of impurity atoms.(1) This can be given by the formula

$$J(|x_1 - x_2|)\varphi(x_1)\varphi(x_2)$$

where $\varphi(x_1)$ and $\varphi(x_2)$ are the spins of impurity atoms, and $J(r) = (k_F r^{-3})\cos(2k_F r)$, where k_F is the Fermi momentum.

The RKKY interaction is long-ranged and rapidly oscillating. In order to investigate it we usually consider models with random Hamiltonians, such as those in which $J(|x_1 - x_2|)$ is assumed to be random and statistically independent for various pairs x_1, x_2.(2,3) Similarly, a lattice model can be used with the Hamiltonian

$$H = \frac{1}{2}\sum_{x_1 \neq x_2} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}} \tag{1}$$

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where $\epsilon(x_1, x_2)$ is random and d is the dimension of the model. The parameter α characterizes the power of the long-range interaction. The main problem we treat here is the existence of the free energy for such systems. For $\alpha > 1$ the answer is in the affirmative.(4,5) In this paper we shall consider values of $\alpha \leqslant 1$.

In the case of nonrandom Hamiltonians the free energy may not exist for such systems. For example, in the ferromagnetic case the energy of the ground state increases more rapidly than the volume. However, in the presence of random interactions all effective interactions decrease. Consequently, the free energy exists and is independent of $\epsilon(x_1, x_2)$ with probability 1.

We assume that $x \in \mathbb{Z}^d$ and that $\varphi(x)$, the spin variable at the point x, takes the values ± 1. Let V be a finite subset of $\mathbb{Z}^d$. The Hamiltonian H in the volume V is given by the expression

$$H(\varphi(V)) = \frac{1}{2} \sum_{\substack{x_1, x_2 \in V \\ x_1 \neq x_2}} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}}$$

where $\varphi(V)$ denotes the spin configuration in V, and $\epsilon(x_1, x_2)$ is a set of independent random variables with a zero mean value defined in the probability space (Ω, σ, P). They are assumed to be translation-invariant, i.e.,

$$\epsilon(x_1 + x, x_2 + x)$$

has the same distribution for all $x \in \mathbb{Z}^d$. For a given realization of the random variables $\omega = \{\epsilon(x_1, x_2), x_1, x_2 \in \mathbb{Z}^d\} \in \Omega$, the free energy in the volume V is defined as follows:

$$f_V(\omega) = \frac{1}{|V|} \log \Xi(V), \qquad \Xi(V) = \sum_{\varphi(V)} \exp[-H(\varphi(V))]$$

where the summation is taken over all the configurations $\varphi(V)$ in the volume V, and $|V|$ is the cardinality of V.

For the sets V we shall choose parellelepipeds tending to infinity in the sense of Fisher.(6) Note that "tending to infinity in the sense of Fisher" means that, if $a_1, a_2, \ldots, a_d$ are the edges of the parallelepipeds and p is its diameter, then

$$a_i/p > \text{const} > 0, \qquad i = 1, \ldots, d \quad \text{as} \quad V \to \infty$$

The main result of this paper is the following theorem.

Theorem. Suppose the following conditions are fulfilled:

I. $\alpha > 1/2$.

IIa. The random variables $\epsilon(x_1, x_2)$ have moments of an arbitrary order, and there exists a constant b such that for all integers $k \geqslant 2$

$$|E(\epsilon^k(x_1, x_2))| \leqslant \tfrac{1}{2} d(x_1, x_2) b^{k-2} \cdot k!, \qquad \forall x_1, x_2 \in \mathbb{Z}^d$$

where $d(x_1, x_2)$ is the variance of $\epsilon(x_1, x_2)$, and $E(\cdot)$ is the averaging over the measure P.

IIb. The variances $d(x_1, x_2)$ are uniformly bounded by some constants C_1 and C_2 so that

$$0 < C_1 \leqslant d(x_1, x_2) \leqslant C_2$$

Then, for any extending sequence of parallelepipeds which tends to infinity in the sense of Fisher, $f_{V_n}(\omega)$ converges to a nonrandom limit with P-probability 1. The limit is independent of the sequence V_n and is equal to the limit of $E(f_{V_n}(\omega))$.

To prove this theorem we use Bernstein's inequality for probabilities of large deviations.[7,8] Let η_i, $i = 1, 2, \ldots$, be independent random variables with the zero mean value satisfying condition IIa of the theorem. Let d_i be the variance of η_i, $D_n = d_1 + \cdots + d_n$. Let $0 < t < \sqrt{D_n}/2b$; then

$$P\left\{\sum_{i=1}^{n} \eta_i \geqslant 2t\sqrt{D_n}\right\} < \exp(-t^2)$$

$$P\left\{\sum_{i=1}^{n} \eta_i \leqslant -2t\sqrt{D_n}\right\} < \exp(-t^2)$$

2. PROOF FOR LIMITED $\epsilon(x_1, x_2)$

Now we prove the theorem for limited $\epsilon(x_1, x_2)$. Suppose the random variables are uniformly bounded, i.e., there exists a constant $C > 0$ such that $|\epsilon(x_1, x_2)| \leqslant C$ with probability 1.

Consider two equal nonoverlapping cubes V^1 and V^2 in the lattice $\mathbb{Z}^d$ with edge of size a. Let z^1 and z^2 be the centers of the cubes V^1 and V^2, respectively.

The energy of the interaction between V^1 and V^2 is given by the expression

$$H(\varphi(V^1), \varphi(V^2)) = \sum_{x_1 \in V^1, x_2 \in V^2} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}}$$

For given $\varphi(V^1)$ and $\varphi(V^2)$, $H(\varphi(V^1), \varphi(V^2))$ depends on $\epsilon(x_1, x_2)$ and is a random variable.

Lemma 1. Let $\operatorname{dist}(V^1, V^2) \geqslant \text{const}\cdot a$. Then for any δ, $0 < \delta < 1/2$ and for all sufficiently large a we have

$$\begin{aligned} P\{\omega \in \Omega:\ & |H(\varphi(V^1), \varphi(V^2))| \geqslant a^{(3/2+\delta)d}/|z^1 - z^2|^{\alpha d}, \\ & \text{at least for one pair } \varphi(V^1), \varphi(V^2)\} \\ & \leqslant \exp(-a^{(1+\delta)d}) \end{aligned} \tag{2}$$

Proof of Lemma 1. Let us fix the configurations $\varphi(V^1)$ and $\varphi(V^2)$ and define the random variables

$$\eta(x_1, x_2) = \epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)|z^1 - z^2|^{\alpha d}/|x_1 - x_2|^{\alpha d}$$

Then

$$H(\varphi(V^1), \varphi(V^2)) = \frac{1}{|z^1 - z^2|^{\alpha d}} \sum_{x_1 \in V^1, x_2 \in V^2} \eta(x_1, x_2)$$

From condition IIa of the theorem it follows that the random variables $\eta(x_1, x_2)$ satisfy the conditions of Bernstein's inequality with the constant $B(V^1, V^2) = \text{const}\cdot b$. Let

$$D(V^1, V^2) = \sum_{x_1 \in V^1, x_2 \in V^2} d\eta(x_1, x_2)$$

where $d\eta(x_1, x_2)$ is the variance of $\eta(x_1, x_2)$. Obviously, $\text{const}\cdot a^{2d} \leqslant D(V^1, V^2) \leqslant \text{const}\cdot a^{2d}$. Suppose $t = a^{(3/2+\delta)d}/2[D(V^1, V^2)]^{1/2}$; then

$$t < [D(V^1, V^2)]^{1/2}/2B(V^1, V^2)$$

for all $0 < \delta < 1/2$ and for sufficiently large a. Therefore, from Bernstein's inequality we have:

$$P\left\{\left|\sum_{x_1 \in V^1, x_2 \in V^2} \eta(x_1, x_2)\right| \geqslant a^{(3/2+\delta)d}\right\} < 2\exp(-t^2)$$

The general number of configuration pairs $\varphi(V^1)$, $\varphi(V^2)$ is equal to $2^{|V^1|+|V^2|}$; hence, the left-hand side of inequality (2) does not exceed $2^{2a^d}\cdot 2\exp(-t^2) < \exp(-a^{(1+\delta)d})$ for sufficiently large a. The Lemma is proven.

Remark. It is not essential that V^1 and V^2 be equal cubes. Lemma 1 and two subsequent Lemmas also hold in the case where V^1 and V^2 are parallelepipeds with edges differing by a constant factor. We shall use this remark below.

Now we estimate $H(\varphi(V^1), \varphi(V^2))$ when V^1 and V^2 are equal adjacent cubes. In Lemma 2 the distance between V^1 and V^2 is not stipulated, although we shall use it in the case in which V^1 and V^2 have a common face.

Lemma 2. There exist γ, $\tau > 0$ such that for all sufficiently large a

$$P\{|H(\varphi(V^1), \varphi(V^2))| \geqslant a^{(1-\gamma)d}, \text{ at least for one pair } \varphi(V^1), \varphi(V^2)\} \leqslant \exp(-a^\tau) \tag{3}$$

Proof of Lemma 2. The proof of Lemma 2 is based on the following inductive statement. Suppose for some s_1, $\tau_1 > 0$ that it has been proven for all sufficiently large a that

$$P\{|H(\varphi(V^1), \varphi(V^2))| \geqslant a^{s_1 d}, \text{ at least for one pair } \varphi(V^1), \varphi(V^2)\} \leqslant \exp(-a^{\tau_1}) \tag{4}$$

Let us fix δ, $0 < \delta < 1/2$; then for

$$s_2 = \min_{0 \leq u \leq 1} \max\left\{s_1 u + (1-u)\left(1 - \frac{1}{d}\right), \left(\frac{3}{2} - \alpha + \delta\right)u + (1-u)(2-\alpha)\right\}$$

and some $\tau_2 > 0$ a similar estimate is valid for all sufficiently large a:

$$P\{|H(\varphi(V^1), \varphi(V^2))| \geqslant \text{const}\cdot a^{s_2 d}, \text{ at least for one pair } \varphi(V^1), \varphi(V^2)\} \leqslant \exp(-a^{\tau_2}) \tag{5}$$

Now we prove this statement. Subdivide V^1 and V^2 into smaller cubes with edge a^1 of the order of a^u, $0 \leqslant u \leqslant 1$. For example, we can take $a^1 = [a^u]$, where $[a^u]$ is all of a^u. The V^1 and V^2 are not to be subdivided evenly into such cubes. Near the boundary there may appear parallelepipeds with smaller edges. These are included into the next subdivision. In this way V^1 and V^2 are subdivided into cubes with edge a^1 and parallelepipeds adjacent to the boundary with edges a_i, $a^1 \leqslant a_i < 2a^1$, $i = 1, \ldots, d$. Let us denote the elements of the subdivisions of V^1 and V^2 by V_i^1 and V_j^2. Then,

$$H(\varphi(V^1), \varphi(V^2)) = \sum_{i,j} H(\varphi(V_i^1), \varphi(V_j^2))$$

The number of the adjacent pairs V_i^1, V_j^2 does not exceed $\text{const}\cdot(a/a^1)^{d-1}$. We use the estimate (4) for these parallelepipeds and apply Lemma 1 to the other pairs V_i^1, V_j^2. We then have

$$P\left\{\text{at least for one pair } \varphi(V^1), \varphi(V^2): |H(\varphi(V^1), \varphi(V^2))| \geqslant \text{const}\cdot(a^1)^{s_1 d}\left(\frac{a}{a^1}\right)^{d-1} + (a^1)^{(3/2+\delta)d} \sum_{i,j} \frac{1}{|z_i^1 - z_j^2|^{\alpha d}}\right\}$$

$$< \text{const}\cdot\left[\left(\frac{a}{a^1}\right)^{d-1} \exp[-(a^1)^{\tau_1}] + \left(\frac{a}{a^1}\right)^{2d} \exp[-(a^1)^{(1+\delta)d}]\right]$$

Here z_i^1 and z_j^2 denote the centers of V_i^1 and V_j^2. Then

$$\sum_{i,j} \frac{1}{|z_i^1 - z_j^2|^{\alpha d}} \leqslant \frac{\text{const}}{(a^1)^{\alpha d}} \left(\frac{a}{a^1}\right)^{(2-\alpha)d}$$

Since $a^1 = [a^u]$, then under the condition of $u \neq 0$ we obtain for sufficiently large a and some $\tau(u) > 0$ the following:

$$P\{\text{at least for one pair } \varphi(V^1), \varphi(V^2): \quad |H(\varphi(V^1), \varphi(V^2))| \geqslant \text{const}\cdot(a^{[s_1 u + (1-u)(1-1/d)]d} + a^{[(3/2-\alpha+\delta)u + (2-\alpha)(1-u)]d})\} \leqslant \exp(-a^{\tau(u)})$$

By taking a u that minimizes the expression $\max\{s_1 u + (1-u)(1-1/d), (\frac{3}{2} - \alpha + \delta)u + (2-\alpha)(1-u)\}$ and noting that this $u \neq 0$, we obtain the inequality (5).

As the first estimate the rough upper bound is taken:

$$\forall \varphi(V^1), \varphi(V^2), \quad |H(\varphi(V^1), \varphi(V^2))| \leqslant \text{const} \cdot a^{(2-\alpha)d}$$

Then, by using the inductive statement in a finite number of steps we obtain the inequality (5) for any $s_2 > \max\{\frac{3}{2} - \alpha + \delta, 1 - 1/d\}$. If $\alpha > 1/2$, we can choose δ so small that $(\frac{3}{2} - \alpha + \delta) < 1$; therefore $\max\{\frac{3}{2} - \alpha + \delta, 1 - 1/d\}$ will be less than unity, which proves Lemma 2.

Lemma 3. For any $\gamma_1 > 0$ and some $\tau(\gamma_1) > 0$, $P\{|H(\varphi(V^1))| \geqslant a^{(1+\gamma_1)d}$, at least for one $\varphi(V^1)\} \leqslant \exp(-a^{\tau(\gamma_1)})$, for sufficiently large a.

The proof of Lemma 3 is analogous to the proof of Lemma 2 and is not presented here.

Let W_k be a cube with edge 2^k on the lattice $\mathbb{Z}^d$. It can be subdivided into 2^d cubes with edge 2^{k-1}. Denote these cubes by W_{k-1}^i, $i = 1, 2, \ldots, 2^d$. Define the random variable g_{W_k} by the equality

$$g_{W_k} = f_{W_k} - \frac{1}{2^d} \sum_{i=1}^{2^d} f_{W_{k-1}^i}$$

Then the following Lemma holds.

Lemma 4. There exists $\gamma_2, \tau_2 > 0$ such that for sufficiently large k

$$P\{|g_{W_k}| \geqslant 1/2^{k\gamma_2}\} \leqslant \exp(-2^{k\tau_2}) \tag{6}$$

Proof of Lemma 4

$$H(\varphi(W_k)) = \sum_{i=1}^{2^d} H(\varphi(W_{k-1}^i)) + \frac{1}{2} \sum_{i \neq j} H(\varphi(W_{k-1}^i), \varphi(W_{k-1}^j)) \tag{7}$$

Applying Lemma 2, we have for some $\gamma_2, \tau_2 > 0$

$$P\{\text{at least for one } \varphi(W_k), \max_{i \neq j} |H(\varphi(W_{k-1}^i), \varphi(W_{k-1}^j))| \geqslant 2^{k(1-\gamma_2)d}\}$$

$$\leqslant \exp(-2^{k\tau_2}) \tag{8}$$

for sufficiently large k. From (7) and (8) we obtain

$$P\left\{\left|f_{W_k} - \frac{1}{2^d} \sum_{i=1}^{2^d} f_{W_{k-1}^i}\right| \geqslant \frac{2^{k(1-\gamma_2)d}}{2^{kd}}\right\} \leqslant \exp(-2^{k\tau_2})$$

and Lemma 4 is proved.

From Lemma 4 it follows that for sufficiently large k, $|E(g_{W_k})| < 2/2^{\gamma_2 k}$, since $|g_{W_k}|$ does not exceed a certain power of the volume of W_k with probability 1. The $f_{W_{k-1}^i}$, $i = 1, 2, \ldots, 2^d$ are independent, identically distributed random variables; therefore, $|Ef_{W_k} - Ef_{W_{k-1}}| = |Eg_{W_k}| < 2/2^{\gamma_2 k}$. Hence, there exists a limit Ef_{W_k} as $k \to \infty$. Denote this limit by F, $F = \lim_{k\to\infty} Ef_{W_k}$. We shall prove that for an initial sequence of parallelepipeds V_n, $f_{V_n} \to F$ as $n \to \infty$ with probability 1.

Consider a squence of subdivisions ξ_k of the lattice $\mathbb{Z}^d$ into cubes with edges 2^k, $k = 1, 2, \dots$, coordinated so that the point $0 \in \mathbb{Z}^d$ is the vertex of these subdivisions for all k.

Let $p_n = \operatorname{diam}(V_n)$. Choose $k(n)$ to satisfy the condition $2^{k(n)} \leqslant p_n{}^l < 2^{k(n)+1}$, $0 < l < 1$. The parameter l depending on α is defined below. $\xi_{k(n)}$ subdivides V_n into cubes with the edge $2^{k(n)}$ and into parallelepipeds of smaller edges adjacent to the boundary of V_n. Similarly, Lemma 2 unites these parallelepipeds with the neighboring cubes, so that we obtain a subdivision of V_n into cubes with edge $2^{k(n)}$, which we denote by $W^i_{k(n)}$, $i = 1, 2, \dots, t(n)$, where $t(n)$ is the number of such cubes, and into parallelepipeds adjacent to the boundary whose edges are less than $2^{k(n)+1}$ and no less than $2^{k(n)}$. We denote these elements of the subdivision by $\overline{W}_n{}^j$, $j = 1, 2, \dots, t'(n)$, where $t'(n)$ is the number of these parallelepipeds. Then we have

$$H(\varphi(V_n)) = \sum_{i=1}^{t(n)} H(\varphi(W^i_{k(n)})) + \sum_{j=1}^{t'(n)} H(\varphi(\overline{W}_n{}^j)) + \sum_{i,j} H(\varphi(W^i_{k(n)}), \varphi(\overline{W}_n{}^j))$$
$$+ \frac{1}{2} \sum_{i \neq i'} H(\varphi(W^i_{k(n)}), \varphi(W^{i'}_{k(n)})) + \frac{1}{2} \sum_{j \neq j'} H(\varphi(\overline{W}_n{}^j), \varphi(\overline{W}^{j'}_n)) \tag{9}$$

For a sufficiently large p_n, when Lemmas 1–3 can be applied to all the elements of the subdivisions $W^i_{k(n)}$, $\overline{W}_n{}^j$, we obtain the following inequality with a probability exceeding $1 - \exp(-p_n^{\tau_3})$, $\tau_3 > 0$:

$$\Bigg| \sum_{i,j} H(\varphi(W^i_{k(n)}), \varphi(\overline{W}_n{}^j))$$
$$+ \frac{1}{2} \sum_{i \neq i'} H(\varphi(W^i_{k(n)}), \varphi(W^{i'}_{k(n)})) + \frac{1}{2} \sum_{j \neq j'} H(\varphi(\overline{W}_n{}^j), \varphi(\overline{W}^{j'}_n)) \Bigg|$$
$$\leqslant \text{const} \cdot (p_n^{(3/2-\alpha+\delta)ld} p_n^{(2-\alpha)(1-l)d} + p_n^{(1-l)d} p_n^{(1-\gamma)ld})$$
$$\left| \sum_{j=1}^{t'(n)} H(\varphi(\overline{W}_n{}^j)) \right| \leqslant \text{const} \cdot p_n^{(1-l)(1-1/d)d} p_n^{(1+\gamma_1)ld}$$

where δ, γ, and γ_1 are defined by Lemmas 1–3 and τ_3 is dependent on δ, γ, γ_1, and l.

By Lemmas 1 and 3 the parameters δ and γ can be taken as small as desired. Therefore, if $\alpha > 1/2$, by choosing δ, γ, and l we find for some $\gamma_3 > 0$ that the following inequality holds with a probability larger than $1 - \exp(-p_n^{\tau_3})$:

$$|H(\varphi(V_n)) - \sum_{i=1}^{t(n)} H(\varphi(W^i_{k(n)}))| \leqslant \text{const}\cdot p_n^{(1-\gamma_3)d} \tag{10}$$

Let us define the random variable g_n:

$$g_n = f_{V_n} - \frac{1}{t(n)} \sum_{i=1}^{t(n)} f_{W^i_{k(n)}}$$

From (9) and (10) we obtain $P\{|g_n| \geqslant \text{const}/p_n^{\gamma_3 d}\} \leqslant \exp(-p_n^{\tau_3})$. This means that $g_n \to 0$ as $n \to \infty$ with probability 1. Indeed, $p_n{}^d > \text{const}\cdot n$, since the sequence V_n is extending and by the Borel–Cantelli Lemma $g_n \to 0$ as $n \to \infty$ with probability 1.

From Lemma 4 and the consequences following from it, we have $Ef_{W_k} \to F$ as $k \to \infty$. For sufficiently large k

$$P\{|g_{W_k}| \geqslant 1/2^{k\gamma_2}\} < \exp(-2^{k\tau_2}), \qquad |Eg_{W_k}| < 2/2^{k\gamma_2} \tag{11}$$

For an arbitrary $\mu > 0$ one can choose k_0 in such a way that $|Ef_{W_{k_0}} - F| < \mu$, $2/2^{k_0\gamma_2} < \mu/2$, and for all $k \geqslant k_0$ the inequalities (11) and the following ones hold:

$$E(g_{W_k} - Eg_{W_k})^2 < 3^2/2^{2k\gamma_2}, \qquad E(g_{W_k} - Eg_{W_k})^4 < 3^4/2^{4k\gamma_2} \tag{12}$$

For sufficiently large p_n, such that $k(n) > k_0$, the cubes $W^i_{k(n)}$ are subdivided into cubes with edge $2^{k(n)-1} - W^i_{k(n)-1}$, $i = 1, 2,\ldots, 2^d t(n)$, then into cubes with edge $2^{k(n)-2} - W^i_{k(n)-2}$, $i = 1, 2,\ldots, 2^{2d}$, and so forth, until a subdivision into cubes with edge $2^{k_0} - W^i_{k_0}$, $i = 1, 2,\ldots, 2^{d(k(n)-k_0)}t(n)$ occurs. Write $t_j(n) = t(n)2^{(k(n)-k_0-j)d}$. By applying Lemma 4 we obtain

$$f_{V_n} = \frac{1}{t_0(n)} \sum_{i=1}^{t_0(n)} f_{W^i_{k_0}} + g^n_{k_0+1} + \cdots + g^n_{k(n)} + g_n \tag{13}$$

where

$$g^n_{k_0+j} = \frac{1}{t_j(n)} \sum_{i=1}^{t_j(n)} g_{W^i_{k_0+j}}, \qquad j = 1, 2,\ldots, k(n) - k_0$$

Note that the $g_{W^i_{k_0+j}}$, $i = 1, 2,\ldots, t_j(n)$ are independent, identically distributed random variables.

For any q, $0 < q < 1$

$$P\left\{\left|\sum_{j=1}^{k(n)-k_0} g^n_{k_0+j}\right| > \frac{\mu}{1-q}\right\} \leqslant \sum_{j=1}^{k(n)-k_0} P\{|g^n_{k_0+j}| > \mu q^{j-1}\}$$

Choose j_n satisfying the condition

$$t_j(n) \geqslant p^{3d/4} \quad \text{if} \quad j < j_n \qquad \text{and} \qquad t_{j_n}(n) < p_n^{3d/4}$$

Then

$$\sum_{j=1}^{k(n)-k_0} P\{|g^n_{k_0+j}| > \mu q^{j-1}\} = \sum_{j=1}^{j_n-1} P\{|g^n_{k_0+j}| > \mu q^{j-1}\} + \sum_{j=j_n}^{k(u)-k_0} P\{|g^n_{k_0+j}| > \mu q^{j-1}\}$$

Let $q = 2^{-\gamma_2}$, where γ_2 is defined by Lemma 4. From the conditions imposed on k_0 it follows that $|Eg_{W_{k_0+j}}| < \frac{1}{2}\mu q^{j-1}$; therefore

$$\sum_{j=1}^{j_n-1} P\{|g_{k_0+j}| > \mu q^{j-1}\} \leqslant \sum_{j=1}^{j_n-1} P\{|g^n_{k_0+j} - Eg_{W_{k_0+j}}| > \tfrac{1}{2}\mu q^{j-1}\}$$

By using Chebychev's inequality we obtain

$$\begin{aligned} P\{|g^n_{k_0+j} &- Eg_{W_{k_0+j}}| > \tfrac{1}{2}\mu q^{j-1}\} \\ &\leqslant \frac{E(g^n_{k_0+j} - Eg_{W_{k_0+j}})^4}{(\frac{1}{2}\mu q^{j-1})^4} \\ &= E\left(\left[\frac{1}{t_j(n)}\sum_{i=1}^{t_j(n)} (g_{W^i_{k_0+j}} - Eg_{W_{k_0+j}})\right]^4\right) \Bigg/ \left(\frac{\mu}{2} q^{j-1}\right)^4 \\ &= \frac{3t_j(n)[t_j(n) - 1][E(g_{W_{k_0+j}} - Eg_{W_{k_0+j}})^2]^2 + t_j(n)E(g_{W_{k_0+j}} - Eg_{W_{k_0+j}})^4}{(t_j(n)\frac{1}{2}\mu q^{j-1})^4} \end{aligned}$$

From (12) it follows that $[E(g_{W_{k_0+j}} - Eg_{W_{k_0+j}})^2]^2 < 3^4 q^{4(j-1)}$ and

$$E(g_{W_{k_0+j}} - Eg_{W_{k_0+j}})^4 < 3^4 q^{4(j-1)}$$

therefore

$$P\left\{|g^n_{k_0+j} - Eg_{W_{k_0+j}}| > \frac{\mu}{2} q^{j-1}\right\} \leqslant \frac{3\cdot 6^4}{t_j^2(n)\mu^4}$$

If $j < j_n$, $t_j(n) \geqslant p_n^{3d/4}$ and since $j_n < \text{const}\cdot\log(p_n)$, we have

$$\sum_{j=1}^{j_n-1} P\{|g^n_{k_0+j}| > \mu q^{j-1}\} \leqslant \frac{j_n\cdot\text{const}}{p_n^{3d/2}\mu^4} \leqslant \text{const}\cdot\frac{\log p_n}{p_n^{3d/2}\mu^4}$$

If $j \geqslant j_n$, $t_j(n) < p_n^{3d/4}$; therefore, $2^{k_0+j} > \text{const}\cdot p_n^{1/4}$ and hence from (11) it follows that

$$\begin{aligned} \sum_{j=j_n}^{k(n)-k_0} P\{|g^n_{k_0+j}| > \mu q^{j-1}\} &\leqslant \sum_{j=j_n}^{k(n)-k_0} t_j(n)\exp(-2^{(k_0+j)\tau_2}) \\ &\leqslant k(n)p_n^{3d/4}\exp(-\text{const}\cdot p_n^{\tau_2/4}) \end{aligned}$$

Thus, we obtain

$$\sum_n P\left\{\left|\sum_{j=1}^{k(n)-k_0} g_{k_0+j}^n\right| > \frac{\mu}{1-q}\right\}$$
$$\leqslant \sum_n \left[\frac{\text{const}\cdot\log p_n}{p_n^{3d/2}\mu^4} + k(n)p_n^{3d/4}\exp(-\text{const}\cdot p_n^{\tau_2/4})\right] < +\infty$$

since $k(n) \leqslant \text{const}\cdot\log(p_n)$, $p_n{}^d \geqslant \text{const}\cdot n$.

This means that with probability 1 for only a finite number of numbers n,

$$\left|\sum_{j=1}^{k(n)-k_0} g_{k_0+j}^n\right| > \frac{\mu}{1-q} \tag{14}$$

By applying the strong law of large numbers we obtain with probability 1 that only for a finite number of numbers n the following relation is satisfied:

$$\left|\frac{1}{t_0(n)}\sum_{i=1}^{t_0(n)} f_{W_{k_0}^i} - Ef_{W_{k_0}}\right| > \mu \tag{15}$$

Moreover, $g_n \to 0$ as $n \to \infty$; therefore $|g_n| > \mu$ only for the finite number of n with probability 1. Now, $|Ef_{W_{k_0}} - F| < \mu$; hence, from (13)–(15) we finally obtain: with probability 1 for all but finitely many n,

$$|f_{V_n} - F| < [3 + 1/(1-q)]\mu$$

Hence, due to the arbitrary choice of μ it follows that $f_{V_n} \to F$ as $n \to \infty$ with probability 1. The theorem is thus proved.

3. THE PROOF OF THE THEOREM IN THE GENERAL CASE AND SOME FURTHER CONSIDERATIONS

In the general case we cannot use the upper bounds which we obtained from the condition of uniform boundedness of the random variables $\epsilon(x_1, x_2)$. To prove Lemma 2 we must perform an initial estimation of $H(\varphi(V^1), \varphi(V^2))$ for the case of adjacent V^1 and V^2. This estimate comes from the following Lemma.

Lemma 5

$$P\left\{\sum_{x_1\in V^1, x_2\in V^2} |\epsilon(x_1, x_2| > \frac{1}{b}\bar{D}(V^1, V^2) + a^{2d2b}\log 2\right\}$$
$$< \exp\left[-\frac{1}{4b^2}\bar{D}(V^1, V^2)\right]$$

where

$$\bar{D}(V^1, V^2) = \sum_{x_1\in V^1, x_2\in V^2} d\epsilon(x_1, x_2)$$

The proof of Lemma 5 is analogous to that of Bernstein's inequality.

In the general case the existence of moments for the random variables f_V is not obvious. Under our conditions f_V has moments of all orders.

Lemma 6

$$|Ef_V| \leqslant \sqrt{C_2}|V|$$

$$E(f_V - Ef_V)^{2k} \leqslant \tfrac{1}{2}|V|^k b^{2k-2} C_2^k (2k)!, \qquad k = 1, 2, \ldots$$

The proof of Lemma 6 is analogous to that of Lemma 2.2 in Ref. 5, taking into account the conditions IIa and IIb of the theorem.

From this Lemma we obtain the estimates (11) and (12). After this the proof of the theorem is analogous to the proof given before.

The problem we considered can be extended to an even more general case. Let the spin variables take values on a compact subset M of space $\mathbb{R}^\nu$. The Hamiltonian is determined by the binary potential $\Phi\colon M \times M \to \mathbb{R}^1$ and is given by

$$H = \frac{1}{2} \sum_{x_1 \neq x_2} \frac{\epsilon(x_1, x_2)\Phi(\varphi(x_1), \varphi(x_2))}{|x_1 - x_2|^{\alpha d}}$$

The potential Φ is assumed to be symmetric, i.e., $\Phi(\varphi_1, \varphi_2) = \Phi(\varphi_2, \varphi_1)$, $\forall \varphi_1, \varphi_2 \in M$. Let a probability measure $d\chi$ be given on M. Then the free energy is defined in a finite volume V by

$$f_V = \frac{\log \Xi(V)}{|V|}; \qquad \Xi(V) = \int_{M^V} \exp[-H(\varphi(V))]\, d\chi^{|V|}(\varphi(V))$$

where $d\chi^{|V|}$ is the direct product of the measure $d\chi$ taken $|V|$ times. The theorem also holds if the potential Φ satisfies Golder's condition with an arbitrary positive exponent, i.e., if there exists $\theta > 0$ and a constant $C_3 > 0$ such that $|\Phi(\varphi_1, \varphi_2) - \Phi(\varphi_1', \varphi_2') \leqslant C_3(|\varphi_1 - \varphi_1'| + |\varphi_2 - \varphi_2'|)^\theta$, $\forall \varphi_1, \varphi_2, \varphi_1', \varphi_2' \in M$.

In conclusion we note that in this problem we are concerned with free energy in a finite volume with empty boundary conditions. If $\alpha > 1/2$, we can consider the existence of free energy with arbitrary boundary conditions. Let V be a finite volume and let $\varphi(\mathbb{Z}^d \backslash V)$ be the spin configuration outside V.

The energy of the configuration $\varphi(V)$ with the boundary conditions $\varphi(\mathbb{Z}^d \backslash V)$ is defined by

$$H(\varphi(V)|\varphi(\mathbb{Z}^d\backslash V)) = \frac{1}{2} \sum_{\substack{x_1, x_2 \in V \\ x_1 \neq x_2}} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}} + \sum_{x_1 \in V} \sum_{x_2 \in \mathbb{Z}^d \backslash V} \frac{\epsilon(x_1, x_2)\varphi(x_1)\varphi(x_2)}{|x_1 - x_2|^{\alpha d}}$$

It is easily seen that, if $\alpha > 1/2$, the second sum converges with probability 1 and $H(\varphi(V)|\varphi(\mathbb{Z}^d\backslash V))$ is definite. We can now define the free energy in a volume V with the boundary conditions $\varphi(\mathbb{Z}^d\backslash V)$ by

$$f_{V;\varphi(\mathbb{Z}^d\backslash V)} = \frac{\log \Xi(V|\varphi(\mathbb{Z}^d\backslash V))}{|V|}$$

$$\Xi(V|\varphi(\mathbb{Z}^d\backslash V)) = \sum_{\varphi(V)} \exp[-H(\varphi(V)|\varphi(\mathbb{Z}^d\backslash V)]$$

The random variables $f_{V_n,\varphi_n(\mathbb{Z}^d\backslash V_n)}$ can be easily shown to converge to F with probability 1 for any sequence of extending parallelepipeds that tend to infinity in the sense of Fisher and for any arbitrary sequence of the boundary conditions $\varphi_n(\mathbb{Z}^d\backslash V_n)$.

Note. The condition $\alpha > 1/2$ is essential. One can show that for $\alpha \leq 1/2$ the free energy of the systems under consideration does not exist, because $f_{V_n} \to +\infty$ as $n \to \infty$ with probability 1.

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